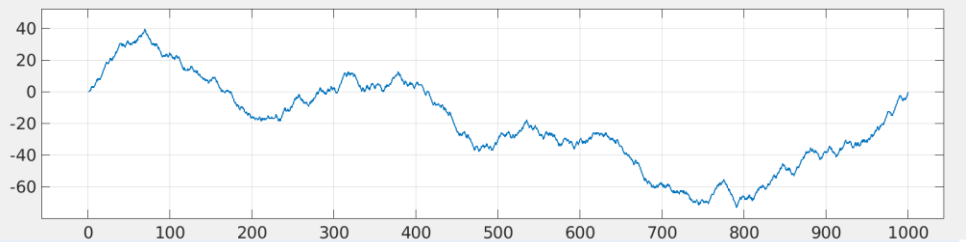
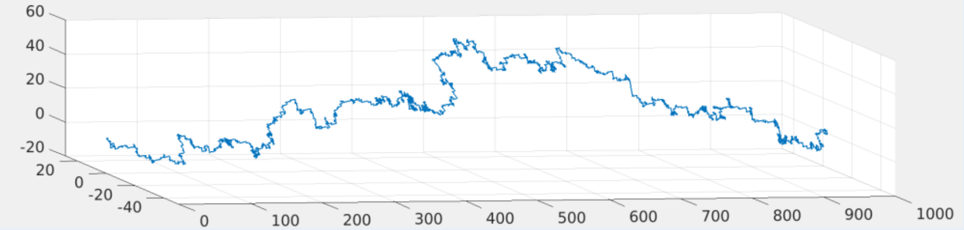
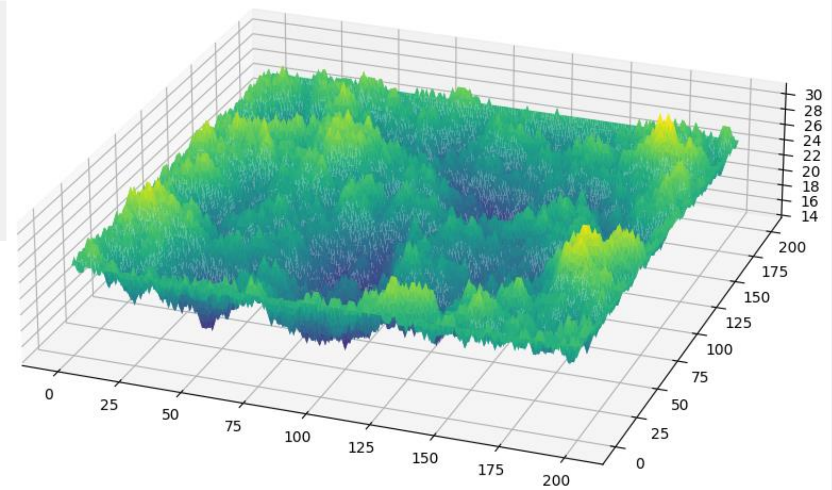
Minimal Surfaces in Random Environment



1-dim. surface in 2-dim. space (d=1, n=1)



1-dim. surface in 3-dim. space (d=1, n=2)



2-dim. surface in 3-dim. space (d=2, n=1)

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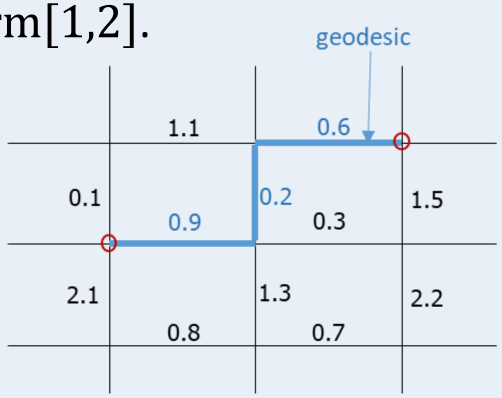
Example 1: First-passage percolation

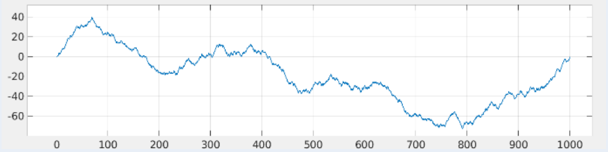
- Idea: Random perturbation of a given geometry, usually formed by a random media with short-range correlations. Initiated by Hammersley–Welsh 1965. We consider the discrete setting of the lattice $\mathbb{Z}^D$.
- Edge weights: Independent and identically distributed non-negative $(\tau_e)_{e\in E(\mathbb{Z}^D)}$. Distribution of τ_e assumed "nice". For instance, $\tau_e \sim \text{Uniform}[1,2]$.
- Passage time: A random metric $T_{u,v}$ on $\mathbb{Z}^D$ given by

$$T_{u,v} := \min \sum_{e\in p} \tau_e$$

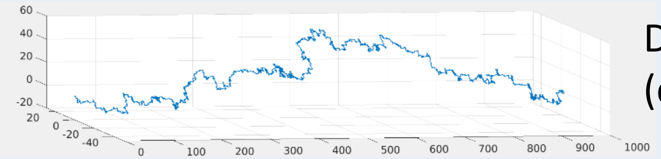
with the minimum over paths p connecting u and v.



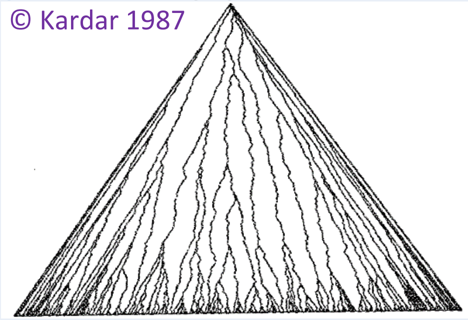
- Geodesic: The unique path p realizing $T_{u,v}$, denoted $\gamma_{u,v}$. Geodesic is a 1-dimensional "minimal surface" in D-dimensional space.
- Goal: Understand the large-scale properties of the metric T. In particular, understand the geometry and length of long geodesics.



D=2
(d=1, n=1)



D=3
(d=1, n=2)



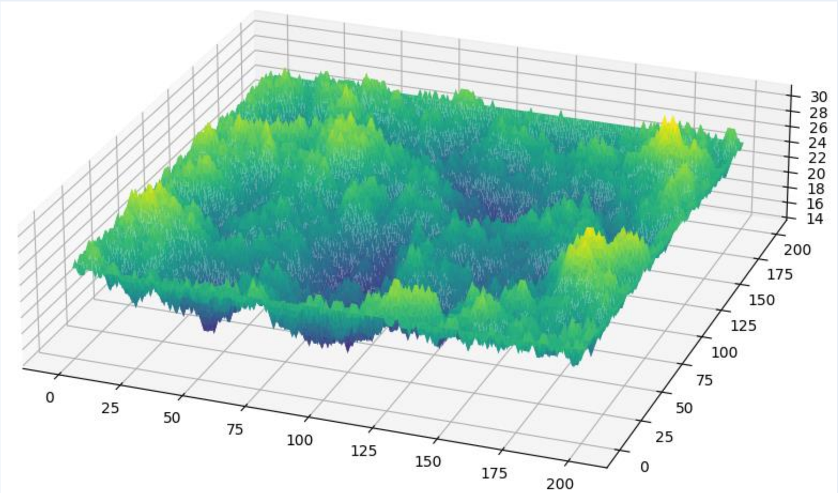
Coalescence of geodesics in D=2 dim.

Example 2: Domain walls in disordered Ising ferromagnet (random-bond Ising model)

- Edge weights (as before): IID non-negative $\tau = (\tau_e)_{e \in E(\mathbb{Z}^D)}$.
  Distribution of τ_e assumed “nice”. For instance, $\tau_e \sim \mathrm{Uniform}[1,2]$.
- Disordered Ising ferromagnet: An Ising model in the “random environment” τ, with the (τ_e) serving as its coupling constants (random-bond Ising model).
  Configurations are $\sigma: \mathbb{Z}^D \to \{-1,1\}$ and the (quenched, formal) Hamiltonian is

$$H^\tau(\sigma) := -\sum_{u \sim v} \tau_{\{u,v\}} \sigma_u \sigma_v$$

- Goal: Understand the geometry and energy of large domain walls of the model at zero temperature (or low temperature).
- Setup: Place the model in a finite cube with Dobrushin boundary conditions. Domain wall forms a $(D-1)$-dimensional “minimal surface” in D dimensions.
- When $D = 2$, domain wall coincides with first-passage percolation geodesic.

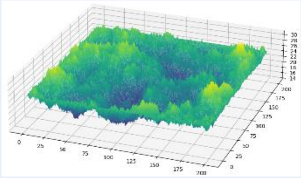


D=3 (d=2, n=1)

Domain walls in disordered Ising ferromagnet – localization and delocalization

- Hamiltonian: $H^{\tau}(\sigma) \coloneqq -\sum_{u\sim v} \tau_{\{u,v\}}\sigma_u\sigma_v$ (random-bond Ising model).
- Setup: Place model in $\Lambda_L \times \mathbb{Z}$ with $\Lambda_L \coloneqq \{-L,\dots,L\}^d$ with Dobrushin boundary conditions. Domain wall forms a $\mathrm{d} = (D-1)$-dimensional "minimal surface" in D dimensions.

D=3
(d=2, n=1)



- Weights: Let $b > a > 0$. Take the weights (τ_e) independent, each distributed as $\mathrm{Uniform}[a,b]$.
- Theorem (Bassan-Dario-P. 25+): The surface delocalizes for $d = 2$ (e.g., expected highest sign change above a uniformly chosen vertex in Λ_L is $\geq c\sqrt{\log\log L}$).
- Theorem (Bassan-Gilboa-P. 23): If $\frac{b-a}{a}$ is small then the surface localizes for $d \geq 3$.
- Bovier–Külske 94,96 previously obtained (non-quantitative versions of) theorems for the disordered Solid-On-Solid approximation (a model with no overhangs).
- Conjecture (Bassan-Gilboa-P. 23. Earlier in physics literature):
  1) $d = 3$: the surface delocalizes when $\frac{b-a}{a}$ is large
  (leading to a roughening transition in the disorder strength in dimension $d = 3$!).
  2) The surface localizes (for all $b > a > 0$) when $d \geq 5$ (possibly also for $d = 4$).

(Harmonic) minimal surfaces in random environment

- Minimal surfaces in random environment (abstract idea):
  d-dimensional surfaces in D=$(d+n)$-dimensional space which minimize the sum of their elastic energy and their environment potential energy, subject to given boundary conditions.
  Of interest in its own right, and related to aforementioned systems.
  We seek a model which is more amenable to analysis!
- Our model: Harmonic minimal surfaces in random environment (harmonic MSRE).
  Configurations are $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ (continuous rather than integer valued!).
  Quenched disorder is $\eta: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ and disorder strength is $\lambda > 0$.
  In a finite domain $\Lambda \subset \mathbb{Z}^d$, the Hamiltonian is

$$H^{\eta,\lambda,\Lambda}(\varphi) := \frac{1}{2} \sum_{\substack{u \sim v \\ \{u,v\} \cap \Lambda \neq \emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v \in \Lambda} \eta_{v,\varphi_v}$$

  The minimal surface $\varphi^{\eta,\lambda,\Lambda,\tau}$ is the configuration minimizing $H^{\eta,\lambda,\Lambda}(\varphi)$ among configurations which coincide with boundary conditions $\tau: \mathbb{Z}^d \to \mathbb{R}^n$ outside Λ.
  (an n-component Gaussian free field in a random environment).
- Goal: Study the geometry and energy of the minimal surface on large domains.

Harmonic minimal surfaces in random environment - background

- Harmonic minimal surfaces in random environment (harmonic MSRE): Configurations are $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ (continuous rather than integer valued!). Quenched disorder is $\eta: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ and disorder strength is $\lambda > 0$. In a finite domain $\Lambda \subset \mathbb{Z}^d$, the Hamiltonian is

$$H^{\eta,\lambda,\Lambda}(\varphi) := \frac{1}{2} \sum_{\substack{u\sim v \\ \{u,v\}\cap\Lambda\neq\emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v\in\Lambda} \eta_{v,\varphi_v}$$

  The minimal surface $\varphi^{\eta,\lambda,\Lambda,\tau}$ is the configuration minimizing $H^{\eta,\lambda,\Lambda}(\varphi)$ among configurations which coincide with boundary conditions $\tau: \mathbb{Z}^d \to \mathbb{R}^n$ outside Λ.
- Mathematics literature: $d = n = 1$: Bakhtin et al. 16-19 in connection to the Burgers equation. On $\mathbb{R}^{d=1}$, n= 1: Bakhtin–Cator–Khanin 14. Related literature on Brownian polymers in random environment – see review by Comets-Cosco 18. Fixed d and $n \to \infty$: Ben-Arous–Bourgade–McKenna 21 (landscape complexity for the Elastic Manifold, following Fyodorov–Le Doussal 20).
- Physics literature (also related models): Huse–Henley 85, Kardar 87, Natterman 87, Middleton 95, Emig–Nattermann 98, Scheidl–Dincer 00, Le Doussal–Wiese–Chauve 04, Husemann–Wiese 18. Reviews: Forgacs–Lipowsky–Nieuwenhuizen 91 (in Domb–Lebowitz vol. 14), Giamarchi 09, Wiese 22.

Harmonic minimal surfaces in random environment – disorder

- Our initial focus is on distributions of the disorder $\eta\colon \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ which are "independent".
- Main example: smoothed white noise, defined as follows:
  - $(\eta_{v,\cdot})_{v\in\mathbb{Z}^d}$ are independent.
  - $\eta_{v,t} = (WN_v * b)(t)$ with WN_v a white noise and b a "bump function" satisfying: (1) $b \geq 0$ and $b(t) = 0$ when $\|t\| \geq 1$, (2) $\int b(t)^2 dt = 1$, (3) b is a Lipschitz function.
- Abstract assumptions (all hold for smoothed white noise):
  - we always assume suitable energy minimizers exist.
  - (stat): for $s\colon \mathbb{Z}^d \to \mathbb{R}^n$, the shifted disorder $\eta^s_{v,t} := \eta_{v,t-s_v}$ has the same distribution as η.
  - (indep): the $(\eta_{v,\cdot})_{v\in\mathbb{Z}^d}$ are independent. For each v, the process $t \mapsto \eta_{v,t}$ is independent at distance 2.
  - (conc): Write $\mathrm{GE}^{\eta,\lambda,\Lambda,\tau} := H^{\eta,\lambda,\Lambda}(\varphi^{\eta,\lambda,\Lambda,\tau})$ for the ground energy. Then for each $\lambda > 0$, $\tau\colon \mathbb{Z}^d \to \mathbb{R}^n$ and finite $\Delta \subset \Lambda \subset \mathbb{Z}^d$, conditioned on $\eta|_{\Delta^c \times \mathbb{R}^n}$ we have that $\mathrm{Std}(\mathrm{GE}^{\eta,\lambda,\Lambda,\tau}) \leq C\lambda\sqrt{|\Delta|}$ with Gaussian tails on this scale.
- Assumptions (stat)+(indep) allow, e.g., to vary disorder strength between vertices.
- For later reference: (stat)+(conc) hold also for periodic disorder.

Localization and delocalization

- We consider the transversal fluctuations of the harmonic MSRE surface on the domain $\Lambda_L := \{-L, -L+1, \ldots, L\}^d$ with zero boundary conditions.
- Theorem (Localization, (stat)+(conc)): There exists $C > 0$, depending only on d, n and the distribution of η, such that for each $v \in \Lambda_L$,

$$\mathbb{E}\left(\left\|\varphi_v^{\eta,\lambda,\Lambda_L}\right\|\right) \le C\sqrt{\lambda}\begin{cases} L^{\frac{4-d}{4}} & d = 1,2,3 \\ \log L & d = 4 \\ 1 & d \ge 5 \end{cases}$$

- Theorem (Delocalization, smoothed white noise): There exists $c > 0$, depending only on the distribution of η and the disorder strength $\lambda > 0$, such that

$$\frac{1}{|\Lambda_L|}\mathbb{E}\left(\left|v \in \Lambda_L : \left\|\varphi_v^{\eta,\lambda,\Lambda_L}\right\| \ge h\right|\right) \ge c$$

with

$$h = \begin{cases} L^{3/5} & d = 1, n = 1 \\ L^{1/2} & d = 1, n \ge 2 \\ L^{\frac{4-d}{4+n}} & d \in \{2,3\} \\ (\log\log L)^{\frac{1}{4+n}} & d = 4 \end{cases}$$

$n = 1$	Lower bound	Predicted	Upper bound
$d = 1$	$L^{0.6}$	$L^{2/3}$	$L^{0.75}$
$d = 2$	$L^{0.4}$	$L^{0.41\pm0.01}$	$L^{0.5}$
$d = 3$	$L^{0.2}$	$L^{0.22\pm0.01}$	$L^{0.25}$
$d = 4$	$(\log\log L)^{0.2}$	$(\log L)^{0.2083\ldots}$	$\log L$
$d \ge 5$	1	1	1

- Physics predictions for $n = 1$:
d=1: Huse–Henley 85, Kardar 85, Huse–Henley–D.S.Fisher 85, Kardar–Parisi–Zhang 86,
d=2,3: Middleton 95, Scheidl–Dincer 00, Le Doussal–Wiese–Chauve 04, Husemann–Wiese 18,
d=4: Emig–Nattermann 98,99.

Scaling relation

- Consider now the harmonic MSRE surface on $\Lambda_L = \{-L, -L+1, \dots, L\}^d$ with zero boundary conditions. It is common in the literature to say that the height fluctuations behave as $L^{\xi_{d,n}}$ while the ground energy fluctuations behave as $L^{\chi_{d,n}}$.
- Scaling relation: It is proposed (e.g., Huse–Henley 85) that, at least for $d \le 4$,

$$\chi_{d,n} = 2\xi_{d,n} + d - 2$$

  We give rigorous versions of this equality for general d, n. Write $\mathrm{Avg}_\Lambda(\cdot)$ for the average operation on Λ. Write $GE^{\eta,\lambda,\Lambda}$ for the energy of the minimal surface.
- Theorem ((stat)+(indep)): There exist $C, c > 0$, depending only on d, such that for all $\mathrm{h} > 0$, all $\lambda > 0$ and unit vector $e \in \mathbb{R}^n$:

  First (version of $\chi_{d,n} \ge 2\xi_{d,n} + d - 2$),

$$\mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda_L} - \mathrm{Med}\left(GE^{\eta,\lambda,\Lambda_L}\right)\right| \ge ch^2 L^{d-2}\right) \ge \frac{1}{3}\mathbb{P}\left(\left|\mathrm{Avg}_{\Lambda_L}\left(\varphi^{\eta,\lambda,\Lambda_L}\right)\cdot e\right| \ge h\right)$$

  Second (version of $\chi_{d,n} \le 2\xi_{d,n} + d - 2$), let $\eta[\Lambda_{\lfloor L/2\rfloor}]$ be η with its middle portion resampled (precisely, $\eta[\Lambda_{\lfloor L/2\rfloor}]$ is obtained by resampling $\eta_{v,\cdot}$ for $v \in \Lambda_{\lfloor L/2\rfloor}$). For $h \ge 1$,

$$\mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda_L} - GE^{\eta[\Lambda_{\lfloor L/2\rfloor}],\lambda,\Lambda_L}\right| \ge Ch^2 L^{d-2}\right) \le C\mathbb{P}\left(\max_{v\in\Lambda_L}\left|\varphi_v^{\eta,\lambda,\Lambda_L}\cdot e\right| \ge h\right)$$

  Third, for $d = 1$: Define $M_k := \max_{L-k\le|v|\le L}\left|\varphi^{\eta,\lambda,\Lambda_L}\cdot e\right|$. Then

$$c\max_{0\le j\le\lceil\log_2 L\rceil} 2^{-j}\left(\mathbb{E}M_{2^j}\right)^2 \le Std\left(GE^{\eta,\lambda,\Lambda_L}\right) \le C\sum_{0\le j\le\lceil\backslash\log_2 L\rceil} 2^{-j}\left(1+\sqrt{\mathbb{E}M_{2^j}^4}\right)$$

Main identity

- The following deterministic identity is our main tool for analyzing the harmonic MSRE model.
- Fix a finite $\Lambda \subset \mathbb{Z}^d$ and the disorder strength $\lambda > 0$. We abbreviate

$$H^\eta(\varphi) := H^{\eta,\lambda,\Lambda}(\varphi) = \frac{1}{2} \sum_{\substack{u\sim v \\ \{u,v\}\cap\Lambda\neq\emptyset}} \|\varphi_u - \varphi_v\|_2^2 + \lambda \sum_{v\in\Lambda} \eta_{v,\varphi_v} = \frac{1}{2}\|\nabla\varphi\|_\Lambda^2 + \lambda \sum_{v\in\Lambda} \eta_{v,\varphi_v}$$

- Lemma (main identity): For each $\varphi: \mathbb{Z}^d \to \mathbb{R}^n$ and $s: \mathbb{Z}^d \to \mathbb{R}^n$ we have

$$H^{\eta^s}(\varphi + s) - H^\eta(\varphi) = (\varphi, -\Delta_\Lambda s) + \frac{1}{2}\|\nabla s\|_\Lambda^2$$

where $(-\Delta_\Lambda s)_v := \sum_{\substack{u:\, u\sim v \\ \{u,v\}\cap\Lambda\neq\emptyset}} (s_v - s_u)$ and the shifted disorder $\eta^s: \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$ is

$$\eta^s_{v,t} := \eta_{v,t-s_v}$$

- Proof: Indeed, the disorder term cancels in the first equality of

$$H^{\eta^s}(\varphi + s) - H^\eta(\varphi) = \frac{1}{2}(\|\nabla(\varphi + s)\|_\Lambda^2 - \|\nabla\varphi\|_\Lambda^2)$$

$$= \frac{1}{2}\Big(\big(\nabla(\varphi + s), \nabla(\varphi + s)\big)_\Lambda - (\nabla\varphi, \nabla\varphi)_\Lambda\Big) = (\nabla\varphi, \nabla s)_\Lambda + \frac{1}{2}(\nabla s, \nabla s)$$

$$= (\varphi, -\Delta_\Lambda s) + \frac{1}{2}\|\nabla s\|_\Lambda^2$$

and a discrete Green's identity is used in the last step.

Localization and $\chi_{d,n} \geq 2\xi_{d,n} + d - 2$ (ideas from proof I)

- Lemma (height and energy): For each $\lambda > 0, \Lambda \subset \mathbb{Z}^d$ finite, $s: \mathbb{Z}^d \to \mathbb{R}^n$ and $r > 0$:

$$\mathbb{P}\left(\left|\left(\varphi^{\eta,\lambda,\Lambda}, -\Delta_\Lambda s\right)\right| \geq r\right) \leq 3 \inf_{\gamma \in \mathbb{R}} \mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda} - \gamma\right| \geq \frac{r^2}{4\|\nabla s\|_\Lambda^2}\right)$$

- Proof: Abbreviate $\varphi^\eta = \varphi^{\eta,\lambda,\Lambda}$ and similarly GE^η. By main identity, for each $\rho \in \mathbb{R}$,

$$H^{\eta^{\rho s}}(\varphi + \rho s) - H^\eta(\varphi) = \rho(\varphi, -\Delta_\Lambda s) + \frac{\rho^2}{2}\|\nabla s\|_\Lambda^2$$

- In particular, with $\rho = -\frac{r}{\|\nabla s\|_\Lambda^2}$,

$$\{(\varphi^\eta, -\Delta_\Lambda s) \geq r\} \subset \left\{H^{\eta^{\rho s}}(\varphi^\eta + \rho s) - H^\eta(\varphi^\eta) \leq -\frac{r^2}{2\|\nabla s\|_\Lambda^2}\right\}$$

$$\subset \left\{\mathrm{GE}^{\eta^{\rho s}} - \mathrm{GE}^\eta \leq -\frac{r^2}{2\|\nabla s\|_\Lambda^2}\right\}$$

- This implies the lemma as $\eta^{\rho s} \overset{d}{=} \eta$ by (stat), and redoing the argument with $-r$.

Localization and $\chi_{d,n} \geq 2\xi_{d,n} + d - 2$ (ideas from proof II)

- Lemma (height and energy): For each $\lambda > 0, \Lambda \subset \mathbb{Z}^d$ finite, $s: \mathbb{Z}^d \to \mathbb{R}^n$ and $r > 0$:
$$\mathbb{P}\left(\left|\left(\varphi^{\eta,\lambda,\Lambda}, -\Delta_\Lambda s\right)\right| \geq r\right) \leq 3 \inf_{\gamma \in \mathbb{R}} \mathbb{P}\left(\left|\mathrm{GE}^{\eta,\lambda,\Lambda} - \gamma\right| \geq \frac{r^2}{4\|\nabla s\|_\Lambda^2}\right)$$
- Application: The version of $\chi_{d,n} \geq 2\xi_{d,n} + d - 2$ given by the inequality
$$\mathbb{P}\left(\left|\mathrm{GE}^{\eta,\lambda,\Lambda_L} - \mathrm{Med}\left(\mathrm{GE}^{\eta,\lambda,\Lambda_L}\right)\right| \geq ch^2 L^{d-2}\right) \geq \frac{1}{3}\mathbb{P}\left(\left|\mathrm{Avg}_{\Lambda_L}\left(\varphi^{\eta,\lambda,\Lambda_L}\right) \cdot e\right| \geq h\right)$$
is the case where $s = 0$ outside Λ_L and $-\Delta_{\Lambda_\mathrm{L}} s = \frac{1}{|\Lambda_L|}$ on Λ_L (so $\|\nabla s\|_{\Lambda_\mathrm{L}}^2 \sim \mathrm{L}^{2-\mathrm{d}}$).
- Concentration assumption (conc): For each $\lambda > 0$ and finite $\Delta \subset \Lambda \subset \mathbb{Z}^d$, conditioned on $\eta|_{\Delta^c \times \mathbb{R}^n}$ we have that $\mathrm{Std}\left(\mathrm{GE}^{\eta,\lambda,\Lambda}\right) \leq C\lambda\sqrt{|\Delta|}$ with Gaussian tails on this scale.
- We thus obtain that the average height on Λ_L is at most of order $\sqrt{\lambda} L^{\frac{4-d}{4}}$.
- To obtain the pointwise localization bounds stated before, we use a multiscale analysis by applying the above lemma with a suitable sequence of functions (s_k).

Delocalization and $\chi_{d,n} \le 2\xi_{d,n} + d - 2$ (ideas from proof I)

- For notational simplicity assume $n = 1$ and omit λ and Λ_L from notation.
- Fix $h \ge 1$. Write $\mathrm{GE}_h^\eta \coloneqq \min_{\varphi:\max_{v\in\Lambda_L}|\varphi_v|\le h} H^\eta(\varphi)$ and let $\varphi^{\eta,h}$ be the minimizer.

  Let $\zeta \coloneqq \eta[\Lambda_{\lfloor L/2\rfloor}]$. Let $A \coloneqq \{GE^\zeta \le GE^\eta - C_0 h^2 L^{d-2}\}$. Let $B^\eta \coloneqq \left\{\max_{v\in\Lambda_L}|\varphi_v^\eta| > h\right\}$.
- Goal (version of $\chi_{d,n} \le 2\xi_{d,n} + d - 2$): $\mathbb{P}(A) \le C\mathbb{P}(B^\eta)$
- Fix $s: \mathbb{Z}^d \to \mathbb{R}$ such that $s = 0$ outside Λ_L and $s = 3h$ inside $\Lambda_{\lfloor L/2\rfloor}$, satisfying $\|\nabla s\|_{\Lambda_L}^2 \approx h^2 \mathrm{L}^{\mathrm{d}-2}$ and $\sum_v |(-\Delta_\Lambda s)_v| \le ChL^{d-2}$.
- Lemma: There exists $\mathrm{C}_1 > 0$ such that $\mathrm{GE}^{\eta^s} \le \mathrm{GE}_h^\eta + C_1 h^2 L^{d-2}$ almost surely.
- Proof: Use main identity: $\mathrm{GE}^{\eta^s} - \mathrm{GE}_h^\eta \le H^{\eta^s}(\varphi^{\eta,h} + s) - H^\eta(\varphi^{\eta,h}) = (\varphi^{\eta,h}, -\Delta_\Lambda s) + \frac{1}{2}\|\nabla s\|_\Lambda^2$.
- Let $A_h \coloneqq \{\mathrm{GE}_h^\zeta \le \mathrm{GE}_h^\eta - C_0 h^2 L^{d-2}\}$ so that $A \subset A_h \cup B^\zeta \cup B^\eta$.
- Let $E_h \coloneqq \{\mathrm{GE}_h^{\eta^s} \le \mathrm{GE}_h^{\zeta^s}\}$ and let $\mathcal{F}$ be the sigma algebra of $(\eta_{v,t})_{v\notin\Lambda_{\lfloor L/2\rfloor},\, t\in\mathbb{R}}$, so that

  (i) $\mathbb{P}(E_h \mid \mathcal{F}) = \frac{1}{2}$ by symmetry, and

  (ii) A_h and E_h are conditionally independent given $\mathcal{F}$ as they are determined by separated disorders.
- Claim: $A_h \cap E_h \subset B^\eta \cup B^{\zeta^s}$ when $C_0 > 2C\_1$. This implies the goal as η, ζ, ζ^s are equally-distributed.
- Proof: The following shows that if $A_h \cap E_h \cap (B^{\zeta^s})^c$ occurs and $C_0 > 2C_1$ then $\mathrm{GE}^\eta < \mathrm{GE}_h^\eta$:

$$\mathrm{GE}^\eta - C_1 h^2 L^{d-2} \overset{Lem}{\le} \mathrm{GE}_h^{\eta^s} \overset{E_h}{\le} \mathrm{GE}_h^{\zeta^s} \overset{(B^{\zeta^s})^c}{=} \mathrm{GE}^{\zeta^s} \overset{Lem}{\le} \mathrm{GE}_h^\zeta + C_1 h^2 L^{d-2} \overset{A_h}{\le} \mathrm{GE}_h^\eta - (C_0 - C_1) h^2 L^{d-2}$$

Delocalization and $\chi_{d,n} \leq 2\xi_{d,n} + d - 2$ (ideas from proof II)

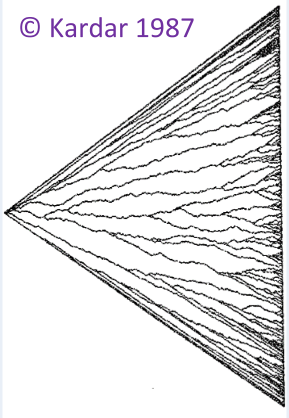
- Delocalization is implied by the scaling inequality (for $h \geq 1$)
$$\mathbb{P}\left(\left|GE^{\eta,\lambda,\Lambda_L} - GE^{\eta[\Lambda_{\lfloor L/2 \rfloor}],\lambda,\Lambda_L}\right| \geq Ch^2L^{d-2}\right) \leq C\mathbb{P}\left(\max_{v\in\Lambda_L}\left|\varphi_v^{\eta,\lambda,\Lambda_L}\cdot e\right| \geq h\right)$$
by giving a lower bound on the fluctuations of $GE^{\eta,\lambda,\Lambda_L} - GE^{\eta[\Lambda_{\lfloor L/2 \rfloor}],\lambda,\Lambda_L}$ on the event that the surface is localized to height h.
- We obtain this by considering the average disorder in $\Lambda_{\lfloor \frac{L}{2} \rfloor} \times [-h,h]^n$.
- The average is Gaussian with standard deviation of order $\lambda\sqrt{\frac{1}{L^d h^n}}$. This implies ground energy fluctuations of order $\lambda\sqrt{\frac{L^d}{h^n}}$ if the surface is localized to height h. Choosing h so that $\lambda\sqrt{\frac{L^d}{h^n}} \geq Ch^2L^{d-2}$ yields delocalization to height $c\lambda^{\frac{2}{4+n}}L^{\frac{4-d}{4+n}}$ (when it is ≥ 1).
- Different lower bound on fluctuations to get $\sqrt{L}$ for $d=1, n\geq 2$.
More work to get "1%" delocalization instead of maximum delocalization.
- The case of $d=4$ dimensions requires a more subtle analysis using fractal (Mandelbrot) percolation, inspired by work of Dario–Harel–P. 21 on the random-field spin $O(n)$ model.

Brief discussion of other disorders I

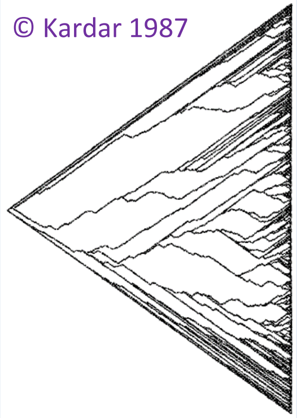
- We are interested in additional options for the disorder $\eta : \mathbb{Z}^d \times \mathbb{R}^n \to (-\infty, \infty]$.
- Brownian disorder (n=1): $t \mapsto \eta_{v,t}$ a two-sided Brownian motion with $\eta_{v,0} := 0$. Provides an approximation to the domain walls of the random-field Ising model.
- Sequel work in preparation (Dembin-Elboim-P.): we consider, more generally, fractional Brownian disorder with Hurst parameter $H \in (0,1)$ and prove that the height fluctuations are exactly of order $L^{\frac{4-d}{4-2H}}$ in dimensions $d = 1,2,3$. We further show that the minimal surface is sub-power-law delocalized for $d = 4$ and localized for $d \geq 5$. These results hold for all values of n.

Geodesics from apex to points on line

Independent disorder



Brownian disorder



Brief discussion of other disorders II

- Periodic disorder: $t \mapsto \eta_{v,t}$ is periodic with respect to the action of $\mathbb{Z}^n$. Stationary to $\mathbb{R}^n$ action. For $n = 1$, provides a "no vortices" approximation to the random-field XY model. Magnetization of the spin model is then in correspondence with localization of the minimal surface. Also describes random-phase Sine-Gordon.
- Our localization results hold also for suitable periodic disorders (those satisfying (stat)+(conc)).
  Thus, our proof that the $d \geq 5$ minimal surface is localized supports the prediction (still open in mathematics) that the random-field XY model retains its ferromagnetic phase at weak disorder and low temperatures in dimensions $d \geq 5$.
- Linear disorder: $\eta_{v,t} = \eta_v \cdot t$. With, e.g., each η_v distributed $N(0,1)$ (much like fractional Brownian motion with Hurst parameter $H = 1$).
  An exactly-solvable case termed random-rod, or Larkin model in physics literature.
- Height fluctuations $L^{\frac{4-d}{2}}$ for $d = 1,2,3$, $\sqrt{\log L}$ for $d = 4$ and localized for $d \geq 5$.
- Integer-valued version: Dario–Harel–P. 2023 prove localization for $d \geq 3$ at weak disorder strength λ.
  Conjecture a roughening transition as disorder strength increases for $d = 3$.

Selected open questions

- Improved exponents: For instance, for $d = 1$ is there a (large) n for which the transversal fluctuations are of order $\sqrt{L}$?
- Periodic disorder (e.g., random-phase sine-Gordon $n = 1$, Giamarchi–Le Doussal 95, Nattermann 90, Orland–Shapir 95, Villain–Fernandez 84):
  $d = 2$: Predictions of "super-roughening" (delocalization to height $\log L$).
  $d = 3$: Delocalization to height $\sqrt{\log L}$.
  Supports power-law magnetization decay prediction for $d = 3$ random-field XY model (Feldman 01, Gingras–Huse 96). What happens in dimension $d = 4$?
- Integer-valued heights (n=1): Is there a roughening transition in the disorder strength in dimension $d = 3$?
  Conjectured in Bassan–Gilboa–P. 23 for domain wall of disordered Ising ferromagnet.
  Conjectured for linear disorder in Dario–Harel–P. 23.
- Shape of the energy and fluctuation distribution:
  Prove unimodality of the distribution and concentration bounds on the scale of its standard deviation.